ON THE STABILITY OF INTEGRATION SCHEMES IN DYNAMIC ANALYSIS OF STRUCTURES

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ABSTRACT

The stability problem in solving numerically the equations of motion of discrete linear structures subjected to dynamic loads is first discussed on a rigorous and complete basis in this paper. A correct stability criterion for the analysis of SDOF system and an equivalence theorem of stability for MDOF system are presented. Secondly, as an example of their applications, the stability of the Newmark method has been studied in detail. Finally, the same problem in dynamic analysis of structures with piecewise linear force-displacement relationships is also discussed in brief. A related stability theorem is provided.

INTRODUCTION

As everyone knows, the dynamic analysis of structures has been playing a great role more and more in the contemporary structural design. And one of the basic works of the analysis is to treat the mathematical model of descrete structures on computers. In linear case it is

$$Ma+Cv+Kd=R \tag{1}$$

where M, C and K are the mass, damping and stiffness matrices respectively of the structures with n degree of freedom and their elements are constants; a, v, and d are the acceleration, velocity and displacement vectors of the MDOF system respectively; R is the load vector. In addition, the damping matrix usually take the Rayleigh form

$$C=\alpha_1 M+\alpha_2 K \tag{2}$$

Here α_1 and α_2 are damping coefficients.

Owing to the explicit reason, solving equations (1) requires the security of stability. (The initial conditions needn't be written in the discussion of this paper.) And many researchers have been paying attention to this problem(1-4). In particular, the authors of Reference(1) fulfiled their discussion in a manner of direct analysis that is more adequate than the others (e.g., the difference form). But refering to some presentations in the monograph (5) without taking notice of the applicable conditions, they introduced a faulty stability criterion in their paper. Having considered the importance of the stability problem, expecially in some engineering practice and research works, such as in dynamic structural identification, and that a perfect solution of this problem has not been got yet, it is necessary that a further research needs to be made carefully.

This paper will present a correct stability criterion for the analysis of SDOF system and an equivalence theorem of stability for MDOF system. As an application of them, the Newmark (γ, β) method is particularly studied and it is proved that the integration scheme of (1) formulated by the method is unconditionally stable if and only if $2\beta \geqslant \gamma \geqslant 1/2$. In addition, according to the necessity of these conditions it is possible to discuss the problem of its conditional stability completely.

The stability problem in nonlinear dynamic analysis of structures is also discussed in brief in this paper. A theoretical result for the stability analysis in computations of elasto-plastic and/or nonlinear elastic dynamic response of structures whose elements are of piecewise linear force-displacement relationships is provided. It demonstrates that an unconditionally stable algorithm in linear case remains uncoditionally stable in these particular nonlinear cases under a certain condition.

FURTHER DISCUSSION OF THE STABILITY IN LINEAR CASE

This section is aimed at making a further research for the general problem of the stability in linear case on a rigorous and complete basis.

First of all equations (1) can be transformed into the conanical form by the modal method: (Rayleigh damping matrix has been assumed).

$$\ddot{X} + \Delta\dot{X} + \Omega^2 X = \bar{M}^{-1}\Phi^T R \qquad (3)$$

where Φ is the transformation matrix of the transformation made

$$d = \Phi X \tag{4}$$

and consists of the mode vectors φ_i (i =1, 2,..., n) of the undamped MDOF system corresponding to (1); M is a diagonal generalized mass matrix whose elements are

$$\overline{m}_i = \varphi_i^T M \varphi_i \qquad (i = 1, 2, \cdots, n) \tag{5}$$

Δ and Ω^2 are both diagonal, whose elements are $2\omega_i \xi_i$ and ω_i^2 (i=1, 2, ..., n), respectively (ω_i, the circle frequency, and ξ_i , the damping ratio, both corresponding to the above modes), and ω_i and ξ_i obey the relations

$$\xi_i = \frac{\alpha_1}{2\omega_i} + \frac{\alpha_2 \omega_i}{2} \qquad (i = 1, 2, \cdots, n) \tag{6}$$

X, $\dot{X}$ and $\ddot{X}$ are the generalized displacement, velocity and acceleration vectors corresponding to the mode matrix Φ .

Now examining one of the uncoupled equations (3), for example, the ith, and after the subscript i is neglected, it is

$$\ddot{x} + 2\omega\xi\dot{x} + \omega^2 x = r \tag{7}$$

where r= $\varphi_i^T R / \overline{m}_i$. This is an equation of motion of SDOF system. There are many numerical procedures for solving equation(7). Here the following integration scheme of the single step direct methods is considered. It is of representative significance for making the stability analysis.

$$\tilde{X}_{t+\Delta t} = A\tilde{X}_t + L r_{t+\Delta t} \tag{8}$$

where $\tilde{X}$ is a vector of components $\ddot{x}$, $\dot{x}$ and x; A is a matrix of order 3 whose elements include the parameters ω , ξ and the integration step Δt and some specific parameters indroduced by the different methods, and A is known as the approximation operator; L is a vector of order 3 whose components also include the above parameters, and it is called the load operator and matters nothing to the stability problem if the load duration is not infinite; t is a certain instant.
From (8) we have

$$\tilde{X}_{t+\Delta t} + \tilde{\varepsilon}_{t+\Delta t} = A(\tilde{X}_t + \tilde{\varepsilon}_t) + L r_{t+\Delta t} \tag{9}$$

where $\tilde{\varepsilon}$ is a vector of components ε'' , ε' and ε which represent the errors of $\ddot{x}$, $\dot{x}$ and x , respectively (including the error of the initial values as well as the round-off). Subtracting (8) from (9) we get

$$\tilde{\varepsilon}_{t+\Delta t} = A\,\tilde{\varepsilon}_t \tag{10}$$

In consequence we have

$$\tilde{\varepsilon}_{t+m\Delta t} = A^m\,\tilde{\varepsilon}_t \tag{11}$$

where m is a positive integer. The meaning of the stability refers to that the finite errors in (10) or (11) at the instant t will not be amplified or will remain bounded at the instant $t+m\Delta t$ after sufficient large m step calcul ations. On the contrary the scheme will be called instable. Therefore the discussion for the stability of some integration scheme comes to examining the behavior of the corresponding A^m for $m \to \infty$. By the way, because the load duration is finite in general, so we have

$$\tilde{x}_{t+(m+j)\Delta t} = A^m [A^j\,\tilde{x}_t + L\,(A^{j-1} r_{t+\Delta t} + \cdots + r_{t+j\Delta t})] \tag{12}$$

where j is a certain number and all the $r_{t+k\Delta t}=0,\ k>j$. And then it can be seen that here the discussion for the bound of $\tilde{\varepsilon}$ is just that for $\tilde{x}$.

Having noticed the related theorems in linear algebra, it can be seen that here A must be similar to a Jordan matrix named its Jordan form. This Jordan matrix is determined uniquely by A except for the ordering of its Jordan submatrices. That is, we have $A = P^{-1}JP$, where J is the above Jordan matrix; P is a transformation matrix. In addition, we also know that the sufficient and necessary coditions making A similar to diagonal matrix are that all its elementary divisors are of degree 1. Thus taking no account of the concrete conditions of A, it is not perfect that Reference (1) recognized J as a diagonal matrix whose elements are the three eigenvalues λ_1, λ_2 and λ_3 of A and consequently took

$$\rho(A) \le 1 \tag{13}$$

where ρ (A) is the spectrum redius of A as the stability criterion to use. (It is easy to know that only in such a case that J has a diagonal matrix form there can be a result that P is composed of the crresponding eigenvectors. In general case the evaluation of P is complicated, but P has no influence on the stability analysis, thereby it needn't be evaluated.) In fact, A includes several parameters as mentioned above. Especially the variation ranges of ω and Δt are very large. Therefore without analysis we cannot put aside the case that the values of these parameters enable A to have an elementary divisor of degree greater than unity. If the modulus of the eigenvalue corresponding to such a divisor is equal to one, then the condition (13) as a stability criterion will be at fault. This point can be seen in the following discussion.

From $A = P^{-1}JP$, we have $A^m = P^{-1}J^mP$. Therefore the original problem is converted into the examination of the behavior of this J^m for $m \to \infty$. There are three phases to be studied:

1. All the elementary divisors of A are of degree 1. In this phase J is a diagonal matrix, i.e.,

$$J = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \tag{14}$$

where there can be the equal among the three eigenvalues λ_1, λ_2 and λ_3. Explicitly, in this phase the stability condition is relation (13).

2. A has a divisor of degree 2. In this phase J is of form

$$J = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & 1 & \lambda_2 \end{bmatrix} \quad \left(or \ J = \begin{bmatrix} \lambda_1 & & \\ 1 & \lambda_1 & \\ & & \lambda_2 \end{bmatrix} \right) \tag{15}$$

where λ_1 and λ_2 can also be the equal. In such a phase we have

$$J^m = \begin{bmatrix} \lambda_1^m & & \\ & \lambda_2^m & \\ & m\lambda_2^{m-1} & \lambda_2^m \end{bmatrix} \quad \left(or \ J^m = \begin{bmatrix} \lambda_1^m & & \\ m\lambda_1^{m-1} & \lambda_1^m & \\ & & \lambda_2^m \end{bmatrix} \right) \tag{16}$$

Thus if $|\lambda_2| \geq |\lambda_1|$, then the stability condition should be

$$\rho(A) < 1 \tag{17}$$

whereas if $|\lambda_2| < |\lambda_1|$, then the condition is the same as relation (13).

3. The elementary divisor of A is an algebraic factor of degree 3. In this phase J becomes

$$J = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix} \tag{18}$$

and we have

$$J^m = \begin{bmatrix} \lambda^m & & \\ m\lambda^{m-1} & \lambda^m & \\ \frac{m(m-1)}{2}\lambda^{m-2} & m\lambda^{m-1} & \lambda^m \end{bmatrix} \tag{19}$$

Consequently the stability condition is the same as (17).

Reviewing the above, we can see that relation (13) is not the sufficient and necessary condition of the stability that Reference (1) mentioned and only is a necessary condition. A sufficient condition can be cited. That is relation (17).

Up to now, we can write down the correct stability criterion of the integration scheme (8).

Theorem 1 (stability criterion). Let the approximation operator of the integration scheme for solving equation (7) be A. The integration scheme will be stable if one of the following three conditions is satisfied.

1) All the elementory divisors of A are of degree 1 and relation (13) holds.

2) Although A has an elementary divisor of degree 2, but the modulus of the eigenvalue corresponding to it is the smaller one and that relation (17) holds.

3) The elementary divisors of A do not belong to the above two cases, but relation (17) holds. nd the integration scheme will be instable if the elementary divisors of A are in any other case.

For convenience of applications an equivalence theorem of the stability for solving equations (1) will be presented as follows.

Theorem 2. Suppose that equation (1) have been transformed into uncoupled equations (3) through relations (2), (5) and (6) and transformation (4). If an integration method with a selected step is stable to all the equations in equations (3), then the method with that step is stable to equations (1). Or else it is instable.

For the proof of this theorem it is enough to notice the following two things. The first one is that the system of the mode vectors φ_i ($i = 1, 2, ..., n$) is linearly independent. Therefore ϕ in transformation (4) is invertible. In addition, there are connections in relations (2), (5) and (6). Consequently equations (1) are equivalent to equations (3). (If the initial conditions are written at the beginning of this paper, the corresponding two initial value problems will be equivalent all the same.) The second is that the integration scheme formulated for (1) and (3) from the same integration method and step are also equivalent dut to the connection of transformation (4). The different point is merely that the latter appears in explicit form already (n matrix operation expressions like (8)), whereas the former is generally a system of linear algebraic equations to be solved for a certain kind of unknown quantities. Thus in the process of integration, the only thing to be done by the former too many by one than the latter is solving the algebraic equations and correspondently the only problem the former has too many by one than the latter is an error in solving the equations. However, such an error is just in the scope that the stability take into account. Hence the results from (1) and (3) are the same as far as the stability of some integration method is concerned.

STABILITY OF THE NEWMARK METHOD

The stability problem of the Newmark (γ, β) method was investigated by many authors and some related results have been provided. For example, the unconditionally stable conditions are $2\beta \geqslant \gamma \geqslant 1/2$ (4). This section will present the same result, but there are two important points in the proof of this paper. The first one is that the proof is based on the correct stability criterion, consequently it is reliable. The second is that from the procedure of the proof we can see that the results presented are not only sufficient conditions but also necessary conditions of the unconditional stability.

Now according to the related theorems in linear algebra studying the Jordan form of A corresponding to the Newmark method for solving equation (7) can be converted into studying that of a matrix B similar to A. We can take

$$B=\begin{bmatrix} -(\frac{1}{2}-\beta)\mu-2(1-\gamma)\nu & -\mu-2\nu & -\mu \\ 1-\gamma-(\frac{1}{2}-\beta)\gamma\mu-2(1-\gamma)\gamma\nu & 1-\gamma\mu-2\gamma\nu & -\gamma\mu \\ \frac{1}{2}-\beta-(\frac{1}{2}-\beta)\beta\mu-2(1-\gamma)\beta\nu & 1-\beta\mu-2\beta\nu & 1-\beta\mu \end{bmatrix} \quad (20)$$

where

$$\mu=\frac{(\omega\Delta t)^2}{1+2\gamma\xi(\omega\Delta t)+\beta(\omega\Delta t)^2} \quad (21)$$

$$\nu=\frac{\xi\mu}{\omega\Delta t}=\frac{\xi(\omega\Delta t)}{1+2\gamma\xi(\omega\Delta t)+\beta(\omega\Delta t)^2} \quad (22)$$

(see Reference (1)). Taking it into account that $\Delta t>0$, $\omega>0$ and $\xi\geqslant 0$, and μ is a monotone function of Δt , it can be seen that the value ranges of μ and ν are $0<\mu<1/\beta$ and $\nu\geqslant 0$, respectively. By way of some operations it is known that B has a unique invarient factor of degree greater than zero. It is

$$d_3(\lambda)=\lambda^3+(\mu+2\nu-2)\lambda^2+(1-2\nu)\lambda \quad (23)$$

(It is interesting that there is not the parameter β in this factor.) Apparently λ is an elementary divisor of B. It corresponds to the eigenvalue $\lambda_1=0$. thereupon the matter which remains to be examined is the factorization of the quadratic expression

$$d_3(\lambda)/\lambda=\lambda^2+(\mu+2\nu-2)\lambda+(1-2\nu) \quad (24)$$

There are three cases to be discussed.

1. If $\mu > [\mu(\gamma+1/2)/2+\nu]^2$, then the other two eigenvalues of B are

$$\lambda_{2,3} = 1-[\mu(\gamma+1/2)/2+\nu] \pm i\{\mu-[\mu(\gamma+1/2)/2+\nu]^2\}^{1/2} \quad (25)$$

And then $|\lambda_{2,3}|=[1-2\nu-(\gamma-1/2)\mu]^{1/2}$. It can be seen that if and only if $\gamma \geqslant 1/2$, the condition $|\lambda_{2,3}| \leq 1$ holds. Notice that this case includes a particular example ($\gamma=1/2$, $\beta=1/4$, $\xi=0$ and $|\lambda_2|=|\lambda_3|=1$) which has been described in Reference (1), where the method of simply calculating the eigenvalues was used. Fortunately there λ_2 and λ_3 correspond respectively to an elementary divisor different from each other. Consequently it does not result in any matter.

2. If $\mu=[\mu(\gamma+1/2)/2+\nu]^2$, then B has an elementary divisor of degree2. It corresponds to a double eigenvalue

$$\lambda_2 = 1-\mu^{1/2} \quad (26)$$

It can be seen that if and only if $\beta \geqslant 1/4$, the condition $|\lambda_2| < 1$ holds.

3. If $\mu < [\mu(\gamma+1/2)/2+\nu]^2$, then the other two eigenvalues of B are

$$\lambda_{2,3} = 1-[\mu(\gamma+1/2)/2+\nu] \pm \{[\mu(\gamma+1/2)/2+\nu]^2-\mu\}^{1/2} \quad (27)$$

In this case the problem depends on λ_3. For $|\lambda_3| \leq 1$ to hold, that is, for

$$[\mu(\gamma+1/2)/2+\nu]+\{[\mu(\gamma+1/2)/2+\nu]^2-\mu\}^{1/2} \leq 2 \quad (28)$$

or

$$(2\beta-\gamma)(\omega\Delta t)^2+4\xi(\gamma-1/2)(\omega\Delta t)+2 \geqslant 0 \quad (29)$$

to hold, we require and only require the inequalities $2\beta \geqslant \gamma \geqslant 1/2$ hold.

Making a summary of the above three cases and according to Theorem 1, we can know first that for any $\Delta t>0$ and any $\omega>0$ as well as any $\xi \geqslant 0$ the Newmark method is unconditionally stable for solving equation (7) if and only if $2\beta \geqslant \gamma \geqslant 1/2$. Next according to Theorem 2 we also can know that Newmark's method is unconditionally stable for equations (1) if and only if

$$2\beta \geqslant \gamma \geqslant 1/2 \quad (30)$$

Because other sufficient conditions of the unconditional stability have ever been got in the course of investigating the Newmark method, now it can be seen that the conditions (30) are already the ones which are impossibly improved due to their necessity.

The investigation for the conditionally stable conditions of the Newmark method is mainly to derive the upper limit formulae. The things we need to consider are in each above case, i.e., when the function

$$f = \mu - [\mu(\gamma+1/2)/2+\nu]^2 \tag{31}$$

or

$$g = \frac{1}{4}[4\beta-(\gamma+\tfrac{1}{2})^2](\omega\Delta t)^2 + \xi(\gamma-\tfrac{1}{2})(\omega\Delta t)+(1-\xi^2) \tag{32}$$

has the different value ranges (greater than, equal to or less than zero), the stability condition is satisfied and at the same time the corresponding unconditionally stable condition is destroyed. (In the third case the alternative or both are destroyed.) After making the overall analysis we can get the complete results. Here we only cite the principal results for practical use (in the case $0 \le \xi < 1$):
For $\gamma < 1/2$,
a) if $4\beta > (\gamma+1/2)^2$ and $0 \le \xi < \xi_0$, then the stable condition is

$$\omega\Delta t < (\omega\Delta t)_3 \tag{33}$$

b) if $4\beta > (\gamma+1/2)^2$ and $\xi_0 \le \xi < 1$, then it is

$$\omega\Delta t < \min[(\omega\Delta t)_2, (\omega\Delta t)_3] \tag{34}$$

c) if $4\beta = (\gamma+1/2)^2$ and $0 \le \xi < 1$, then it is

$$\omega\Delta t < \min[(\omega\Delta t)_3, (\omega\Delta t)_4] \tag{35}$$

d) if $4\beta < (\gamma+1/2)^2$ and $0 \le \xi < 1$, then it is

$$\omega\Delta t < \min[(\omega\Delta t)_2, (\omega\Delta t)_3] \tag{36}$$

where

$$\xi_0 = \{[4\beta-(\gamma+1/2)^2]/2(2\beta-\gamma)]\}^{1/2} \tag{37}$$

$$(\omega\Delta t)_2 = \frac{\xi(\frac{1}{2}-\gamma) - \{2\xi^2(2\beta-\gamma)-[4\beta-(\gamma+\frac{1}{2})^2]\}^{1/2}}{\frac{1}{2}[4\beta-(\gamma+\frac{1}{2})^2]} \tag{38}$$

$$(\omega\Delta t)_3 = 2\xi/(\tfrac{1}{2}-\gamma) \tag{39}$$

$$(\omega\Delta t)_4 = (1-\xi^2)/[\xi(\tfrac{1}{2}-\gamma)] \tag{40}$$

The stability problem encountered in nonlinear dynamic analysis of strucutres is a very difficult one in most cases. However, in some particular cases we may possibly get a few theoretical results. Here a problem in which the structural elements are of piecewise linear force-displacement relationships is discussed.

Under this kind of relationship a theoretical result can be provided as follows.

Theorem 3. An unconditionally stable integration method for the dynamic analysis of linear MDOF structures remains unconditionally stable when it is applied in the elasto-plastic or non-linear elastic dynamic analysis of MDOF structures whose elements are of piecewise linear force-displacement relationships if the number of times of change of the working phase of the structures is finite in the whole response history.

The proof of this theorem and some related problems will be reported in detail in another paper.

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